

LATTICE POINT COVERINGS

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ABSTRACT. We give a simple proof of a necessary and sufficient condition under which any congruent copy of a given ellipsoid contains an integral point and study a similar problem for a 3-dimensional parallelepiped.

1. INTRODUCTION

Motivated by questions originating in simultaneous Diophantine approximation problems, in 1967 I. Niven and H.S. Zuckerman studied lattice point coverings by plane figures [NZ67]. Given a plane convex set S they were interested in necessary and sufficient conditions such that S contains in any position, i.e., with respect to arbitrary translations and rotations, a lattice point of the integral lattice \mathbb{Z}^2 . Their first theorem deals with ellipses and they proved that an ellipse with semi-axes a and b contains a lattice point in any position if and only if $4a^2b^2 \geq a^2 + b^2$.

Here we give a simple proof of a generalisation of their criterion to all dimensions.

Theorem 1.1. *Let $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid with semi-axes α_i , $1 \leq i \leq n$. The following statements are equivalent:*

- i) \mathcal{E} contains a lattice point of \mathbb{Z}^n in any position,
- ii) $\sum_{i=1}^n \frac{1}{\alpha_i^2} \leq 4$,
- iii) \mathcal{E} contains a cube of edge length 1.

To put the statement in a more algebraic form let an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ be given by all points satisfying the inequality

$$(1.1) \quad (x - c)^\top A (x - c) \leq 1,$$

where $A \in \mathbb{R}^{n \times n}$ is a positive definite symmetric matrix and $c \in \mathbb{R}^n$. Let $\text{tr}(\cdot)$ denotes the trace of a matrix. Then we have $\text{tr}(A) = \sum_{i=1}^n 1/\alpha_i^2$ and Theorem 1.1 ii) says that (1.1) has an integral solution for any choice of c and for any rotation of the coordinate system if and only if $\text{tr}(A) \leq 4$.

We remark that the equivalence of the statements i) and ii) in the theorem above follows already from a more general result of Banaszczyk [Ban90b] in the context of connected subgroups of nuclear spaces and balancing vector problems (c.f. [Ban90a], [Ban93], [BS97], [Gia97]). Here, however, we are only interested in the integral lattice \mathbb{Z}^n , and in this case a more simple and geometric proof of the equivalence is available.

Despite ellipses Niven and Zuckermann also studied triangles, rectangles and parallelograms (cf. [NZ67], [NZ69], [Mai69]). Depending on the length of the sides or the distance between opposite sides they derived necessary and sufficient conditions. It seems to be rather hard to generalise these results to arbitrary dimensions in full strength, i.e., e.g., for a rectangular box one would like to have tight necessary and sufficient conditions depending on the length of its sides (cf. [BS97]). Indeed, even in the case of a cube we do not know the minimum edge length such that a cube contains a lattice point in any position. It is easy to see that in terms of the inradius a parallelogram has the lattice point property if its inradius is not less than $1/\sqrt{2}$. In fact, this bound is also best possible as the diamond centred at $(1/2, 1/2)^\top$, i.e.,

$$(1.2) \quad (1/2, 1/2)^\top + \{x \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1\}$$

shows. In the 3-dimensional case we have the following result.

Theorem 1.2. *A 3-dimensional parallelepiped $P \subset \mathbb{R}^3$ contains a lattice point of \mathbb{Z}^n in any position if its inradius is not less than $1/\sqrt{2}$, and this bound is tight.*

In particular, the theorem and the example (1.2) above imply that the minimum edge length of a 3-dimensional cube containing a lattice point in any position is $\sqrt{2}$.

2. PROOFS

First we need some elementary definitions and facts from geometry of numbers for which we refer to [GL87]. A lattice Λ is the set of all integral combinations of n linearly independent vectors $b_i \in \mathbb{R}^n$, $1 \leq i \leq n$, or equivalently a lattice is a set of the form $\Lambda = B\mathbb{Z}^n$, where $B \in \mathbb{R}^{n \times n}$ is a regular matrix. Let \mathcal{B}^n be the n -dimensional unit ball centred at the origin, and let the Euclidean norm be denoted by $\|\cdot\|$. The inhomogenous minimum $\mu(\Lambda)$ of a lattice measures the maximum (Euclidean) distance between a point $x \in \mathbb{R}^n$ and the lattice, i.e.,

$$\mu(\Lambda) = \sup_{x \in \mathbb{R}^n} \inf_{b \in \Lambda} \|x - b\|.$$

By definition we have that for any $c \in \mathbb{R}^n$ the set $c + \mu(\Lambda)\mathcal{B}^n$ contains a lattice point of Λ . The next lemma gives an upper bound on $\mu(\Lambda)$ in terms of the length of linearly independent vectors contained in Λ . It is somehow folklore (cf. [GLS93, pp. 149]), but for sake of completeness we include its short proof.

Lemma 2.1. *Let $\Lambda \subset \mathbb{R}^n$ be a lattice and let $a_1, \dots, a_n \in \Lambda$ be linearly independent. Then*

$$\mu(\Lambda) \leq \frac{1}{2} \left(\|a_1\|^2 + \dots + \|a_n\|^2 \right)^{\frac{1}{2}}.$$

Proof. We proceed by induction on the dimension. For $n = 1$ the statement is certainly true. Hence let $n \geq 2$ and let $x \in \mathbb{R}^n$ with $x = \sum_{i=1}^n \lambda_i a_i$ for some $\lambda_i \in \mathbb{R}$. Next we find a lattice point in Λ which is “close” to x . To this end let L be the linear space generated by a_1, \dots, a_{n-1} and let \bar{x} be the orthogonal projection of x onto the affine plane $[\lambda_n]a_n + L$, where $[\lambda]$ denotes a nearest integer of a real λ . Then we have

$$(2.1) \quad \|x - \bar{x}\| \leq |\lambda_n - [\lambda_n]| \|a_n\| \leq \frac{1}{2} \|a_n\|.$$

$\bar{x} - [\lambda_n]a_n$ belongs to L and it is easy to see that $\Lambda \cap L$ is (or may be identified with) an $(n-1)$ -dimensional lattice. Thus, by our inductive argument, there exists a lattice point $\tilde{b} \in \Lambda \cap L$ such that

$$\|\bar{x} - [\lambda_n]a_n - \tilde{b}\|^2 \leq \frac{1}{4} (\|a_1\|^2 + \dots + \|a_{n-1}\|^2).$$

With $b = [\lambda_n]a_n + \tilde{b} \in \Lambda$ and (2.1) we find

$$\|x - b\|^2 = \|x - \bar{x}\|^2 + \|\bar{x} - b\|^2 \leq \frac{1}{4} (\|a_1\|^2 + \dots + \|a_n\|^2),$$

which gives the desired inequality. \square

Now we come to the proof of the first theorem.

Proof of Theorem 1.1. First we show the equivalence of i) and ii). To see that ii) is necessary let $\rho < \sqrt{n}/2$ and let \mathcal{E} be the ball of radius ρ centred at the $(1/2, \dots, 1/2)^\top$. Then we have $\alpha_i = \rho$, $\sum_{i=1}^n 1/\alpha_i^2 < 4$ and $\mathcal{E} \cap \mathbb{Z}^n = \emptyset$.

Now let $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid satisfying ii) and let $D = \text{diag}(\alpha_1, \dots, \alpha_n)$ be the diagonal matrix with entries α_i . Let $V = (v_{ij}) \in \mathbb{R}^{n \times n}$ be a suitable orthonormal matrix such that $\mathcal{E} = c + V D \mathcal{B}^n$, where c denotes the centre of \mathcal{E} . Obviously,

$$\mathcal{E} \cap \mathbb{Z}^n \neq \emptyset \Leftrightarrow ((V D)^{-1}c + \mathcal{B}^n) \cap (V D)^{-1}\mathbb{Z}^n \neq \emptyset,$$

and in order to verify the right hand side we have to show

$$(2.2) \quad \mu(\Lambda) \leq 1,$$

where $\Lambda = (V D)^{-1}\mathbb{Z}^n = \text{diag}(1/\alpha_1, \dots, 1/\alpha_n) V^\top \mathbb{Z}^n$. Let $a_i \in \Lambda$, $1 \leq i \leq n$, be the column vectors of the matrix $(V D)^{-1}$ then we get

$$\sum_{i=1}^n \|a_i\|^2 = \sum_{i=1}^n \sum_{j=1}^n \frac{1}{\alpha_j^2} v_{ij}^2 = \sum_{j=1}^n \sum_{i=1}^n \frac{1}{\alpha_j^2} v_{ij}^2 = \sum_{j=1}^n \frac{1}{\alpha_j^2}.$$

Together with Lemma 2.1 and ii) we obtain

$$\mu(\Lambda) \leq \frac{1}{2} (\|a_1\|^2 + \dots + \|a_n\|^2)^{\frac{1}{2}} = \frac{1}{2} \left(\sum_{j=1}^n \frac{1}{\alpha_j^2} \right)^{\frac{1}{2}} \leq 1.$$

Thus (2.2) is verified and so the equivalence i) \Leftrightarrow ii).

In order to show that ii) is equivalent to iii) we may assume that

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n \frac{x_i^2}{(\alpha_i)^2} \leq 1 \right\}.$$

If ii) holds then \mathcal{E} contains certainly the cube $C = [-1/2, 1/2]^n$ of edge length one with vertices $(\pm 1/2, \dots, \pm 1/2)^\top$. So it remains to show that iii) implies ii) and to this end let \bar{C} be a cube of edge length 1 contained in \mathcal{E} . With a suitable orthonormal matrix $V = (v_{ij}) \in \mathbb{R}^{n \times n}$ and a vector $t \in \mathbb{R}^n$ we may write $\bar{C} = t + VC$. By the symmetry of C and \mathcal{E} we have $VC \subset \mathcal{E}$ and hence

$$\sum_{i=1}^n \frac{1}{(\alpha_i)^2} \left(\sum_{j=1}^n v_{ij} \epsilon_j \frac{1}{2} \right)^2 \leq 1,$$

for all sign vectors $\epsilon \in \{-1, 1\}^n$. Summing up over all vertices leads to

$$\begin{aligned} 2^n &\geq \frac{1}{4} \sum_{i=1}^n \frac{1}{\alpha_i^2} \sum_{\epsilon \in \{-1, 1\}^n} \left(\sum_{j=1}^n v_{ij} \epsilon_j \right)^2 \\ &= \frac{1}{4} \sum_{i=1}^n \frac{1}{\alpha_i^2} \sum_{\epsilon \in \{-1, 1\}^n} \sum_{j=1}^n (v_{ij} \epsilon_j)^2 = \frac{1}{4} 2^n \left(\sum_{i=1}^n \frac{1}{\alpha_i^2} \right). \end{aligned}$$

Hence we get ii). □

Next we give the proof of Theorem 1.2.

Proof of Theorem 1.2. The bound is certainly tight as already the 2-dimensional example (1.2) shows. So let $u_i \in \mathbb{R}^3$, $\|u_i\| = 1$, $\nu_i \in \mathbb{R}$, $1 \leq i \leq 3$, and

$$P = \{x \in \mathbb{R}^3 : |u_i^\top x| \leq \nu_i, 1 \leq i \leq 3\}$$

be a parallelepiped with centre 0 and inradius at least $1/\sqrt{2}$, i.e., $\nu_i \geq 1/\sqrt{2}$, $1 \leq i \leq 3$. For $c \in \mathbb{R}^3$ we have to show that $(c+P) \cap \mathbb{Z}^3 \neq \emptyset$. To this end let $z_c \in \mathbb{Z}^3$ be a best approximation of c with respect to the maximum norm, i.e., for $p := z_c - c$ we have $|p_i| \leq 1/2$, $1 \leq i \leq 3$. Moreover let $\epsilon_i \in \{-1, 1\}$ be the signs of the coordinates p_i and we set

$$a_0 := p, \quad a_i := p - \epsilon_i e_i, \quad 1 \leq i \leq 3,$$

where e_i denotes the i -th unit vector. Obviously, in order to verify $(c+P) \cap \mathbb{Z}^3 \neq \emptyset$ it suffices to prove that

$$(2.3) \quad P \cap \{a_0, a_1, a_2, a_3\} \neq \emptyset.$$

Suppose the contrary. Then for each a_i exists an u_{j_i} such that a_i violates u_{j_i} , i.e., $|u_{j_i}^\top a_i| > 1/\sqrt{2}$. Hence w.l.o.g. we may assume that u_1 is violated by two points a_j and a_k with $j < k$. Now we distinguish two cases.

First we assume that a_j and a_k differ in exactly one coordinate, i.e., we have $j = 0$ and w.l.o.g. let $u_1^\top a_0 > 1/\sqrt{2}$ and let $k = 1$. Since $\|a_0 - a_1\| = 1 \leq \sqrt{2}$ we also must have $u_1^\top a_1 > 1/\sqrt{2}$. Since the first coordinate of a_0 and a_1

have different signs and since the remaining coordinates coincide we can find a $\lambda \in [0, 1]$ such that $a := \lambda a_0 + (1 - \lambda)a_1 = (0, p_2, p_3)^\top$. Hence, on account of $|p_i| \leq 1/2$ we get the contradiction

$$1/\sqrt{2} \geq \|a\| \geq |u_1^\top a| = \lambda u_1^\top a_0 + (1 - \lambda) u_1^\top a_1 > 1/\sqrt{2}.$$

Thus we are left with the case that p_j and p_k differ in two coordinates. W.l.o.g. let $j = 1, k = 2$ and $u_1^\top a_1 > 1/\sqrt{2}$. Since $\|a_1 - a_2\| = \sqrt{2}$ we also have $u_1^\top a_2 > 1/\sqrt{2}$. Let

$$\lambda := \frac{1 + \epsilon_1 p_1 - \epsilon_2 p_2}{2}$$

and

$$a := \lambda a_1 + (1 - \lambda) a_2 = \left(\frac{p_1 + \epsilon_1 \epsilon_2 p_2 - \epsilon_1}{2}, \frac{p_2 + \epsilon_1 \epsilon_2 p_1 - \epsilon_2}{2}, p_3 \right)^\top,$$

and for abbreviation we set $\beta = \epsilon_1 p_1 + \epsilon_2 p_2$. Then we may write $a = (\epsilon_1(\beta - 1)/2, \epsilon_2(\beta - 1)/2, p_3)^\top$. Since $\lambda \in [0, 1]$ we have $u_1^\top a > 1/\sqrt{2}$ and so

$$\frac{1}{2} < \|a\|^2 = 2 \left(\frac{\beta - 1}{2} \right)^2 + (p_3)^2.$$

With $|p_3| \leq 1/2$ we obtain the quadratic inequality

$$\beta^2 - 2\beta + 1/2 > 0$$

with roots $1 \pm 1/\sqrt{2}$. Since $0 \leq \beta \leq 1$ we get $\beta < 1 - 1/\sqrt{2}$ and thus

$$\|a_0\|^2 = (p_1)^2 + (p_2)^2 + (p_3)^2 \leq \beta^2 + (p_3)^2 < \left(1 - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{4} < \frac{1}{2}.$$

The inradius of the parallelepiped P , however, is at least $1/\sqrt{2}$ and thus the last inequality shows $a_0 \in P$ which contradicts our assumption. \square

3. REMARKS

Finally we would like to pose the following problem mentioned already in the introduction: For the n -dimensional unit cube C_n determine the minimum number γ_n such that $\gamma_n C_n$ contains a lattice point (of the integral lattice \mathbb{Z}^n) in any position.

By Theorem 1.2 we have $\gamma_2 = \gamma_3 = \sqrt{2}$. To the best of our knowledge no further exact values of γ_n are known. A much more general result of Banaszczyk and Szarek [BS97] implies in particular that $\gamma_n = O(\sqrt{\log n})$ as n tends to infinity. Moreover, if the so called Komlós conjecture were true then as an immediate consequence we would know that γ_n is bounded by an universal constant (cf. [BS97], [Gia97]).

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